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Prediction of the fracture toughness of fibrous composites

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A statistical/micromechanical model is developed for the prediction of the fracture toughness of fibrous composites. The fracture resistance of the material is assumed to be related to the statistical distribution of the fiber pull-out length. The distribution of the fiber pull-out length is derived from the fiber strength distribution. The *R*-curve behavior of the fibrous composite is predicted and interpreted based on the present model. The limiting fracture toughness is predicted to be proportional to the square root of the ineffective length, or proportional to the square root of the fiber length if the fiber length is less than the ineffective length.

1. Introduction

During crack growth, several failure mechanisms can be activated in fibrous composites. These mechanisms include fiber breakage, matrix damage, fiber/matrix debonding, fiber pull-out, etc. Even a brittle-fiber/brittle-matrix composite can have a high fracture toughness. For example, a brittle epoxy resin with fracture energy of about 10^2 J m^{-2} can be combined with brittle glass fibers to form a composite system which may have fracture energy of up to 10^5 J m^{-2} [1]. Such large increase in fracture energy is attributed to the above-mentioned failure mechanisms. Like ductile metals, the fibrous composite also exhibits the so-called *R*-curve behavior [2], i.e. the energy required for catastropic or large-scale crack extension is larger than that for the initiation or small-scale crack growth. The *R*-curve behavior has been explained with some success based on the crack bridging concept [3–6]. In particular, a comprehensive account on the crack bridging concept has been given in Cotterell and Mai [5]. In this paper, an alternative viewpoint is proposed. It is believed that an alternative viewpoint may be important for an independent verification of the results. In the crack-bridging model, the relationship between the fiber restraining force and crack opening displacement in the crack wake can not be easily determined or verified. On the other hand, the material parameters appeared in the present prediction model (as shown in the following sections) can be measured in the post-mortem specimens. It should also be noted that both the anisotropy of materials and the blunt crack tip are taken into account. These factors have been ignored in the above-mentioned models.

The model composite is unidirectionally reinforced with fibers. The external tensile loading is applied parallel to the fiber direction, while the crack is assumed to be perpendicular to the fibers. Both the fibers and matrix are brittle, so that the toughness of the individual phase is negligible. The debonding process of the fiber from the matrix is assumed to occur before the fiber breaks. Thus the main crack growth resistance comes from the fiber pull-out mechanism. It should be noted that the theoretical results can be applied to the situation that fibers are randomly oriented, provided that some precautions are observed.

It has been experimentally confirmed that the pull-out length of fibers is not a constant [7]. This implies that the resistance of the composite material to crack growth is varied from one place to another. The present theoretical analysis is based on this observation. Accordingly, a statistical approach is adopted. The distribution of the pull-out lengths is derived from the fiber strength distribution. The crack tip is no longer to be treated as sharp as in the case of homogeneous brittle materials. The crack blunting due to the presence of fibers is taken into consideration. The criterion for crack growth is assumed to be that the stress near the rounded crack tip must exceed some threshold value. The *R*-curve behavior of fibrous composites can thus be explained consistently. It is interesting to compare the present micromechanical theory with the phenomenlogical theory developed earlier [8]. Both theories are aimed at explaining and predicting the stable crack growth behavior of the material based on similar concepts.

2. The stress field ahead of a blunt crack

The stress field around a blunt crack has been studied by several investigators [9, 10]. Specifically, the stress at the blunt crack tip is given by

$$\sigma = R\frac{K}{\sqrt{2\pi\rho}} \tag{1}$$

where K is the nominal stress intensity factor; ρ is the radius of the curvature of the crack tip; R is the stress rounding factor. In general, R is dependent on the actual profile near the crack tip [11]. Furthermore, R is also dependent on the anisotropy of the material [12]. For example, when the material is isotropic and the crack is idealized as a slender ellipse then $R = 2\sqrt{2}$. On the other hand, in an orthotropic material, the value of R depends on the ratio of the Young's moduli along two principal directions E_1 to E_2 and its determination requires the actual crack profile. Nevertheless, for most fibrous composites the range of the ratio of E_1 to E_2 is 10^{-2}–10^2. Accordingly the range of R is restricted by $2 < R < 9$ for most fibrous composites.

3. Micromechanical model

It is assumed that for crack extension the crack must fully open, i.e. the strengthening fiber must be pulled out partially or completely. Accordingly, if the local pull-out length is ℓ, it is assumed that the crack tip opening displacement and the radius of the rounded crack tip are about this same value. Consequently, we take $\rho \approx \ell$. Therefore the stress near the crack tip is given by

$$\sigma = \omega \frac{K}{\sqrt{2\pi\ell}} \tag{2}$$

where ω is a geometrical factor accounting for the stress rounding factor R and the approximation of $\rho \approx \ell$. Furthermore, it is assumed that the crack growth is possible only when σ attains a critical value σ_c. In general, σ_c can be taken the strength of the fiber. In terms of the nominal stress intensity factor, this implies that

$$K \geq S\sqrt{2\pi\ell} \tag{3}$$

is the condition for crack growth, where S denotes σ_c/ω. Since ℓ is not a constant, a statistical distribution about ℓ must be established.

4. Distribution of pull-out lengths

The statistical distribution of fiber pull-out lengths has been studied [7, 13, 14, 6]. Since some incorrect arguments made in [13], a new theoretical analysis is presented here.

Typical crack tip region is sketched in Fig. 1 where ℓ_d and ℓ_c denote the debond length and ineffective length respectively. Before the fiber breaks, the stress variation along the fiber $\sigma(y)$ is governed by the equilibrium condition and the stress transfer between the matrix and fiber. Several stress transfer models have been proposed. In elastic stress transfer cases, Rosen [15] indicated

$$\ell_c \sim d\left(\frac{E_f}{G_m}\right)^{1/2}\left(\frac{1 - V_f^{1/2}}{V_f^{1/2}}\right)^{1/2} \tag{4}$$

where d is the fiber diameter, E_f the Young's modulus of the fiber, G_m the shear modulus of the matrix, V_f the volume fraction of the fiber.

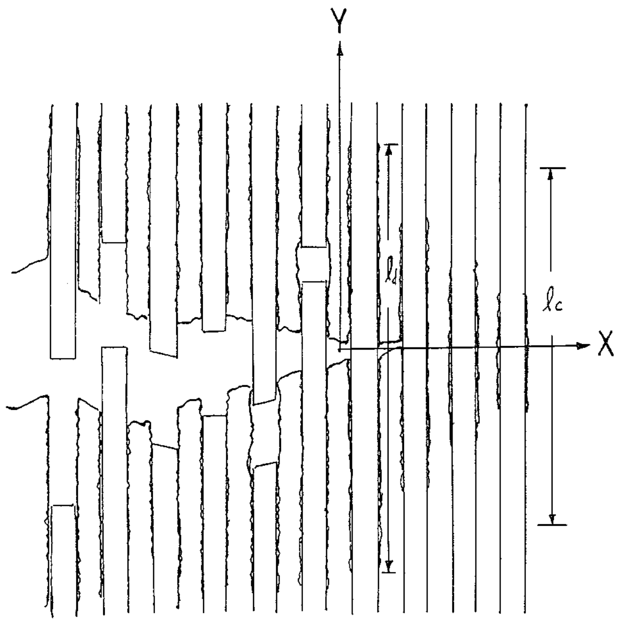

Figure 1 Schematic diagram of crack tip region in a fibrous composite.

Based on the finite difference solution, Termonia [16] found

$$\ell_c \sim d\frac{E_f}{E_m} \tag{5}$$

In "plastic" stress transfer cases, a typical estimation is given by Kelly [17]

$$\ell_c = d\frac{\langle\sigma_f\rangle}{2\langle\tau\rangle} \tag{6}$$

where $\langle\sigma_f\rangle$ and $\langle\tau\rangle$ are the average fiber strength and fiber/matrix interface shear stress. Here we have somehow modified the original model, since there are large scatters in the fiber strength. The strength variation of the fiber is usually given in terms of the Weibull distribution, i.e.

$$P(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^{\alpha}\right] \tag{7}$$

where σ_0 and α are constants.

Within the ineffective length, i.e. $y \leq \ell_c/2$ (refering to Fig. 1), we expect that the single-site fiber breakage probability is much greater than the multi-sites fiber breakage probability. Hence if we divide $\ell_c/2$ into n cells of equal length as shown in Fig. 2. The frequency of the fiber breaks at y_i is

$$f_i \sim P(\sigma(y_i))\prod_{j\neq i}^{n}(1 - P(\sigma(y_j))) \tag{8}$$

Accordingly, the relative frequency is

$$\frac{f_i}{f_j} = \frac{P_i/(1 - P_i)}{P_j/(1 - P_j)} \tag{9}$$

where $P_i \equiv P(\sigma(y_i))$. Passing to the limit, we find that the cumulative distribution for the fiber pull-out length

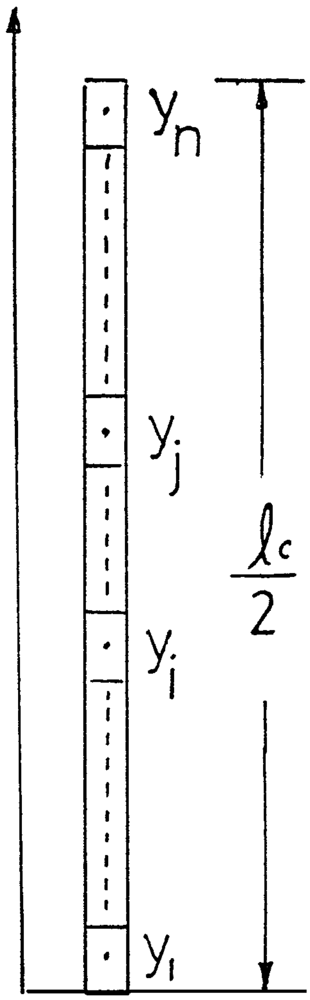

Figure 2 A segment of fiber of length $\ell_c/2$ being divided into n cells.

is given by

$$F(y) = \int_0^y \frac{P(\sigma(\xi))}{1 - P(\sigma(\xi))}\, d\xi \Bigg/ \int_0^{\ell_c/2} \frac{P(\sigma(\xi))}{1 - P(\sigma(\xi))}\, d\xi \tag{10}$$

or

$$F(y) = \left[\int_0^y \exp\left(\frac{\sigma(\xi)}{\sigma_0}\right)^\alpha d\xi + y \right] \Bigg/ \left[\int_0^{\ell_c/2} \exp\left(\frac{\sigma(\xi)}{\sigma_0}\right)^\alpha d\xi + \frac{\ell_c}{2} \right] \tag{11}$$

Thus, once the stress variation along the fiber $\sigma(y)$ is known, $F(y)$ can be directly computed by (11). A special case is explored here. Before fiber breaking, if the fiber is completely detached from the matrix, then $\sigma(y)$ is a constant. It is concluded that

$$F(y) = \frac{2y}{\ell_c} \quad y \le \frac{\ell_c}{2} \tag{12}$$

The distribution is independent on strength variation of the fiber. Since in a debonded fiber $\sigma(y)$ decreases as y increases, the prediction of Equation 11 usually would yield a concave curve as shown in Fig.3. While the distribution (12) only corresponds to some extreme conditions, the experimental data [7] can nevertheless

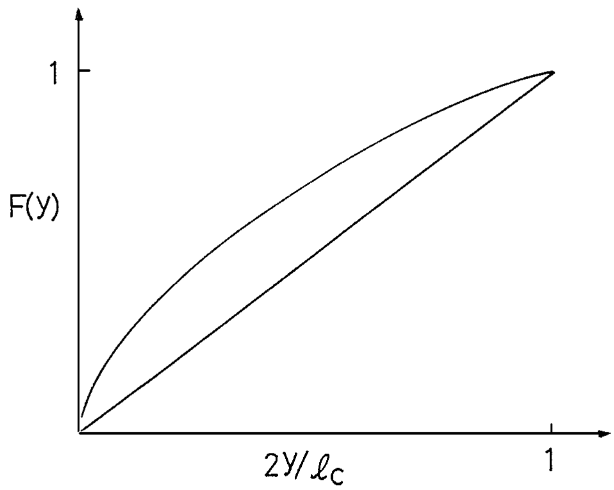

Figure 3 Typical statistical distributions of the pull-out lengths.

be reasonably fitted by this equation, except at the lower tail. This can be easily explained by the following arguments. Since the usual procedure for measuring the pull-out length is to measure the length of the fiber extruding out the crack surface, so the experimental data are usually larger than those defined here by a small amount δ due to the irregular structure of the crack surface. Thus, we shall use the following statistical distribution for pull-out lengths

$$F(y) = \begin{cases} 0 & y < \delta \\ \dfrac{2y}{\ell_c} & \delta \le y \le \dfrac{\ell_c}{2} \\ 1 & y > \dfrac{\ell_c}{2} \end{cases} \tag{13}$$

According to Equation 3, the local resistance to crack growth is proportional to $\sqrt{\ell}$. Therefore, the probability of pulling out one fiber (or one strand of fibers) near the crack tip by the driving "force" K is assumed to be

$$F_1 \equiv \text{Prob}(1; K) = \begin{cases} 0 & K < S\sqrt{2\pi\delta} \\ \left(\dfrac{K}{S\sqrt{\pi\ell_c}}\right)^2 & S\sqrt{2\pi\delta} \le K < S\sqrt{\pi\ell_c} \\ 1 & S\sqrt{\pi\ell_c} \le K \end{cases} \tag{14}$$

If the crack growth distance is not large compared with the original crack length, then the nominal stress intensity factor K is almost a constant during crack growth. Consequently, the probability of pulling out m fibers is given by

$$F_m \equiv \text{Prob}(m; K) = (F_1)^m = \begin{cases} 0 & K < S\sqrt{2\pi\delta} \\ \left(\dfrac{K}{S\sqrt{\pi\ell_c}}\right)^{2m} & S\sqrt{2\pi\delta} \le K < S\sqrt{\pi\ell_c} \\ 1 & S\sqrt{\pi\ell_c} \le K \end{cases} \tag{15}$$

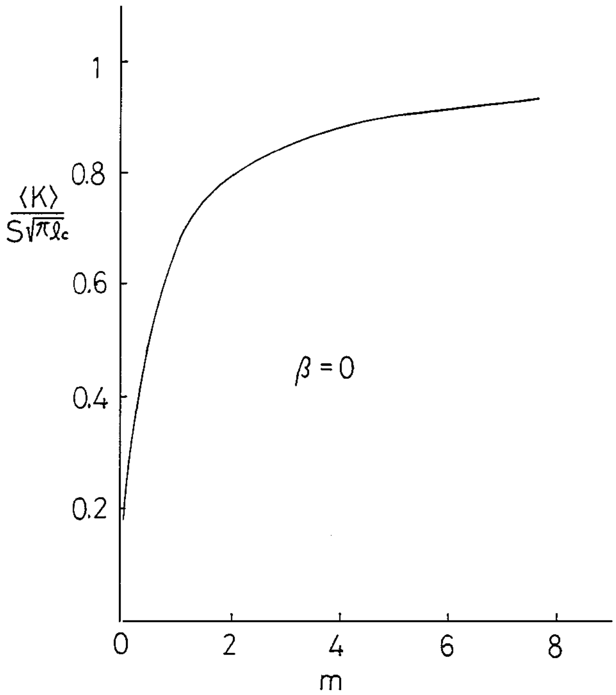

Figure 4 The predicted crack growth resistance as a function of growth distance.

Accordingly, the average $\langle K \rangle$ to drive a crack to propagate a fixed distance $m\lambda$ where λ is the fiber spacing can be determined by integrating the whole possible range of K for constant m, i.e.

$$\begin{aligned}\langle K \rangle &= \int_{S\sqrt{2\pi\delta}}^{S\sqrt{\pi\ell_c}} K \frac{\partial F_m}{\partial K} \mathrm{d}K \\ &= \frac{2m}{2m+1}(S\sqrt{\pi\ell_c} - S\sqrt{2\pi\delta}) \end{aligned} \tag{16}$$

or

$$\frac{\langle K \rangle}{S\sqrt{\pi\ell_c}} = \frac{2m}{2m+1}(1 - \sqrt{\beta}) \tag{17}$$

where $\beta \equiv 2\delta/\ell_c$. In Fig. 4, a schematical curve is ploted by using the continuous version of Equation 17, i.e. m is taken to be a real variable. The R-curve behavior of the material is vividly seen. Furthermore, the standard deviation can also be computed,

$$\frac{\hat{\sigma}}{S\sqrt{\pi\ell_c}} = \frac{1}{2m+1}\sqrt{\frac{m}{m+1}(1-\beta)} \tag{18}$$

Generally speaking, some part of the toughness is independent on growth distance, which is associated with the inherent fracture toughness of the fiber and the matrix. For the fibrous composites investigated in this paper the inherent part of the toughness is much less than the contribution from the pull-out mechanism. Nevertheless, for generality we shall modify Equation 17 by adding a term independent on the growth distance, so

$$\frac{\langle K \rangle}{S\sqrt{\pi\ell_c}} = (1 - \sqrt{\beta})\frac{2m}{2m+1} + \gamma \tag{19}$$

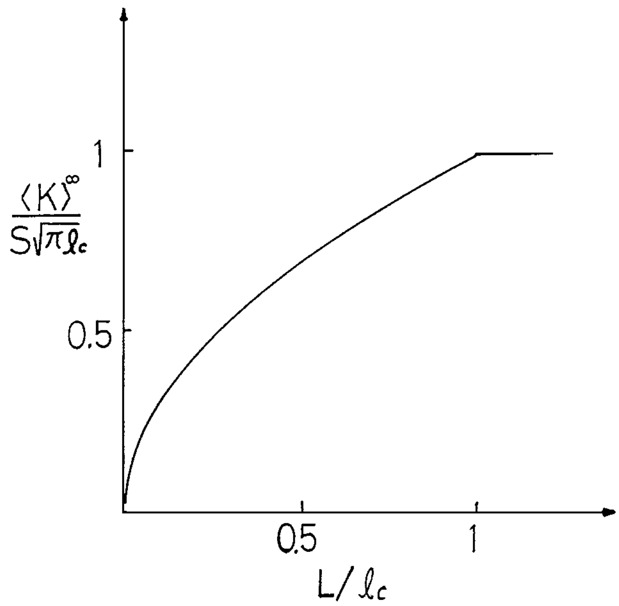

Figure 5 The predicted limiting fracture toughness as a function of the fiber length.

The fracture toughness of the large scale crack growth (the limiting fracture toughness $\langle K \rangle^\infty$) can be found by assuming m is large. Hence from Equation 19 we obtain

$$\frac{\langle K \rangle^\infty}{S\sqrt{\pi\ell_c}} = (1 - \sqrt{\beta}) + \gamma \tag{20}$$

When the contributions of β and γ are negligible, $\langle K \rangle^\infty$ is roughly about $S\sqrt{\pi\ell_c}$. For short fibrous composites with the average fiber length L smaller than ℓ_c, the limiting fracture toughness is predicted by $S\sqrt{\pi L}$. However, it should be noted that for short-fiber reinforced composites, dut to the presence of fiber ends, the fiber may be pulled out before breaking if the embedded length is not long enough [18]. A more rigorous treatment should take this matter into accounts. In this paper we shall ignore this issue. Accordingly, the relationship between the limiting fracture toughness and the fiber length is predicted as shown in Fig. 5 by assuming S is a constant. Since the experimental data reveal that the average fiber strength decrease as the fiber length increases, it may be infered that the limiting toughness actually would attain its maximum when the fiber length is equal to the ineffective length. The conclusion is supported by the experimental results [19].

5. The influence of orientational distribution of fibers

In the foregoing analysis, it is assumed that the fibers are all aligned. Practically, different lay-ups must be used for different structural applications. Experimental results [19] indicates that the pull-out length distribution is relatively independent on the fiber orientation. Hence we may assume that ℓ_c remains unchanged. Thus the limiting fracture toughness is expected to be proportioal to the composite strength in the loading direction, if the pull-out mechanism dominates the failure proess. In other words, the orientational dependence of the limiting fracture toughness can be shown [18] as

$$\langle K(\theta)\rangle^{\infty} = \langle K(\theta = 0)\rangle^{\infty} \cos^2\theta \tag{21}$$

where θ is the misaligned angle between the loading and the fiber direction. On the other hand, it should be noted that, for instance, in a cross-ply laminate the 90°-ply apparently contribute little to the strength of the composite, but its presence usually would assist the debonding process of fibers in the 0°-ply. Therefore, we should not underestimate the importance of the weaker layer in enhancing the toughness of the composite.

For 2D randomly distributed fiber reinforced composites, approximately we have

$$\langle K(2R)\rangle^{\infty} = \frac{1}{2}\langle K(\theta = 0)\rangle^{\infty} \tag{22}$$

For 3D random structure composites, we find

$$\langle K(3R)\rangle^{\infty} = \frac{1}{3}\langle K(\theta = 0)\rangle^{\infty} \tag{23}$$

6. Limiting fracture toughness

In this section the limiting fracture toughness of some typical fibrous composites are calculated. The critical stress σ_c is assumed to be the average fiber strength $\langle\sigma_f\rangle$. The fiber volume fraction is assumed to be 0.65. The geometrical factor ω is then estimated as 2.5. The diameter of the fiber is assumed to be 10 μm. The tensile strengths of the glass and graphite fibers are 3.1 GPa and 3.44 GPa respectively. The shear strengths of the epoxy and polyester resins are 30 MPa and 25 MPa respectively. The estimation of ℓ_c is then based on Equation 6 and is confirmed with the experimental data [7]. The final results are shown in Table I. The theoretical predictions are not in perfect agreement with the experimental data, but the order of magnitude is in agreement. Specifically, in the present model only the fiber pull-out mechanism contributes to its toughness while in a real material system other failure mechanisms always be present and can contribute to its toughness. Despite that there are some uncertainties about the actual material properties used in experiments, such a fair agreement indicates that the fiber pull-out is the dominant failure mechanism in these fibrous composites.

The theoretical results indicate a way to produce fibrous composites with both high strength and high fracture toughness. Fibers of smaller diameter should be used and strong binding agents are used to glue them into fiber strands of high strength. On the other hand, in order to have high toughness (to increase the ineffective length) the binding materials of low strength should be used to produce the final composite materials by binding these strong fiber strands together.

TABLE I The limiting fracture toughness

Material	ℓ_c (mm)	Exp.	$\langle K(0)\rangle^{\infty}$ (MPa $\sqrt{\text{m}}$)	$\langle K(2R)\rangle^{\infty}$ (M Pa $\sqrt{\text{m}}$)	K_Q Exp.
Glass/epoxy	0.52	0.6[a]	50	25	22[b]
Glass/polyester	0.62	1.4[a]	55	28	28[b]
Graphite/epoxy	0.57	1[a]	58	29	44[c]

[a]Ref[7].
[b]Ref[2].
[c]Ref[20].

7. Concluding remarks

During crack growth, the fracture resistance of the fibrous composites is varied along the crack path. This variation can be infered from the statistical variation of fiber pull-out lengths. The crack tip configuration can be described in terms of the local pull-out length. Accordingly the stochastic resistance of the material can be related to the statistical description of the pull-out length. Furthermore, we also present a mathematical analysis to derive the pull-out length distribution from the fiber strength distribution. Based on the above mentioned procedures, a statistical/micromechanical model is developed for the prediction and explanation of the R-curve behavior of fibrous composites. The limiting fracture toughness is found to be proportional to the square root of the ineffective length, as the maximal pull-out length can reach up to one half of the ineffective length, or proportional to the square root of the fiber length if the fiber length is less than the ineffective length.

It should be noted that the limiting fracture toughness is only an indicator for the maximal potential of the material's resistance to crack growth. Under usual conditions, we may not be able to fully realize this potential. Hence, only some fraction, say F%, of this value should be used for the design purpose. It is evident that for the materials with the same limiting fracture toughness, the value of F% may have to be chosen differently. In fact the value of F% depends critically on the fiber strength variation, particularly on the α in Equation 7. For the same level of reliability against fracture, the larger the α value the smaller the F% value should be used. Since larger α implies the more homogeneous in the fiber strength, this situation in turn indicates the pull-out length distribution would be concentrated around a very short interval.

Finally, it should be pointed out that the present analysis is based on the assumption that the debonding process occurs first, and the fiber breakage follows. The pull-out process is then responsible for resisting crack growth. Under certain circumstances, the debonding and pull-out processes may be inhibited by the strong interface strength, or by the high loading rates [21], the dramatical decrease in the pull-out length implies a large decrease in the fracture toughness. For a comprehensive treatment on the prediction of the fracture toughness of fibrous composites, the role played by the interface strength must be properly considered. Such a treatment is beyond the scope of the present article. Interested readers can find the recent and past developments in this field in Ref [22].

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